

# SPECIAL LAGRANGIAN TORI ON A BORCEA-VOISIN THREEFOLD

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Borcea-Voisin threefolds are Calabi-Yau manifolds. They are constructed by Borcea ([B]) and Voisin ([V]) in the construction of mirror manifolds. In [SYZ], Strominger, Yau and Zaslow propose a geometric construction of mirror manifolds using special Lagrangian tori (called SYZ-construction below). Using degenerate Calabi-Yau metrics Gross and Wilson show that SYZ-construction works for any Borcea-Voisin threefolds ([GW]). In this short note we show that there are special Lagrangian tori on one family of Borcea-Voisin threefolds with respect to non-degenerate Calabi-Yau metrics. These tori cover a large part of the threefolds and are perturbations of the special Lagrangian tori used by Gross and Wilson in [GW]. The method is studying the degeneration of Calabi-Yau metrics using gluing (type I degeneration). There are other examples of special Lagrangian torus in compact Calabi-Yau threefolds. Bryant has a beautiful construction of the special Lagrangian torus in some quintic threefolds (see [Br]). In section 2 we give a family of special Lagrangian submanifolds which cover  $K_{\mathbb{C}P^n}$ .

In some sense this note is a continuation of [L]. In that paper we show the existence of immersed special Lagrangian tori in a Kummer type threefold. In this paper we show the existence of embedded special Lagrangian tori in one family of Borcea-Voisin threefolds. Readers may refer to that paper for basic definitions if necessary.

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## 1. SPECIAL LAGRANGIAN TORI ON BORCEA-VOISIN THREEFOLDS

Let  $E_i$  be an elliptic curve with periods 1 and  $\tau_i$  ( $i = 1, 2, 3$ ). The Borcea-Voisin Threefold in discussion is the resolution of quotient  $E_1 \times E_2 \times E_3$  by  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . We denote the quotient by  $M_0$ . Let  $z_1, z_2, z_3$  be the coordinates of the three elliptic curves. The generators of the two  $\mathbb{Z}_2$  actions are

$$\begin{aligned}\alpha : z_1 &\rightarrow -z_1 + \frac{1}{2}, \quad z_2 \rightarrow -z_2 + \frac{1}{2}, \quad z_3 \rightarrow z_3, \\ \beta : z_1 &\rightarrow -z_1, \quad z_2 \rightarrow z_2, \quad z_3 \rightarrow -z_3.\end{aligned}$$

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

The fixed locus of  $\alpha$  is the union of sixteen elliptic curves:

$$\left(\frac{2 + \tau_1 \pm 1 \pm \tau_1}{4}, \frac{2 + \tau_2 \pm 1 \pm \tau_2}{4}\right) \times E_3.$$

The fixed locus of  $\beta$  is the union of sixteen elliptic curves:

$$\frac{1 + \tau_1 \pm 1 \pm \tau_1}{4} \times E_2 \times \frac{1 + \tau_3 \pm 1 \pm \tau_3}{4}$$

The fixed locus of  $\alpha \circ \beta$  is empty. Note that the 32 elliptic curves in the fixed locus do not intersect each other and their images in  $M_0$  are 16 elliptic curves. Denote the image set by  $F$ .

Let  $\pi : M \rightarrow M_0$  be the resolution map by a single blowup of  $F$ . Then  $M$  is a Borcea-Voisin threefold. To see this fact, first we resolve the quotient of  $E_1 \times E_2 \times E_3$  by  $\alpha$ . According to Kummer construction, we get  $K \times E_3$ , where  $K$  is a Kummer surface. Then  $\beta$  induces an action on  $K \times E_3$ . This is exactly the action used in Borcea-Voisin threefold construction (see [B] or [V]).  $M$  is the Borcea-Voisin threefold constructed from  $K \times E_3$  and  $\beta$ . Note that the fixed locus of the involution on  $K$  is two tori and  $M$  is self mirror.

**1a.** Holomorphic (3,0)-form on  $M$ . Note that  $dz_1 \wedge dz_2 \wedge dz_3$  is a holomorphic (3,0)-form on  $E_1 \times E_2 \times E_3$ . Let  $\Omega_0$  be the induced holomorphic (3,0)-form on  $M_0$ . Denote  $\pi^*\Omega_0$  by  $\Omega$ . Then

**Lemma 1.**  $\Omega$  is a holomorphic (3,0)-form on  $M$ .

*Proof.* We only need to check that  $\pi^*\Omega_0$  extends across the exceptional divisors and is nonzero everywhere on the exceptional divisors. Since  $M$  is resolved by a single blow-up along the singular elliptic curves in  $M_0$ . In the normal direction the singularities are of the form  $\mathbb{C}^2/\mathbb{Z}_2$ . We know the required extension is possible for  $\mathbb{C}^2/\mathbb{Z}_2$  (see, for example, [L], the proof of Lemma 3.1).  $\square$

**1b.** Ricci-flat metrics on  $M$ . First we describe the Ricci-flat metrics on the total spaces of the normal bundles of the exceptional divisors in  $M$ . There are two types of exceptional divisors in  $M$ : 8 copies of  $\mathbb{C}P^1 \times E_3$  and 8 copies of  $\mathbb{C}P^1 \times E_2$ . Let  $E$  be either  $E_2$  or  $E_3$ . The total space of the normal bundle of  $\mathbb{C}P^1 \times E$  in  $M$  is  $K_{\mathbb{C}P^1} \times E$ . We can identify  $K_{\mathbb{C}P^1} \times E$  with the resolution of  $(\mathbb{C}^2/\mathbb{Z}_2) \times E$ . Let  $w_1, w_2$  be coordinates on  $\mathbb{C}^2$  and  $w_3$  be a coordinate on  $E$ . Define  $U = |w_1|^2 + |w_2|^2$  and

$$(1) \quad f_a(U) = U \sqrt{1 + \frac{a^2}{U^2}} + a \ln \frac{U}{\sqrt{U^2 + a^2} + a}, \quad a > 0.$$

Then  $\tilde{g}_a = \sqrt{-1} \partial \bar{\partial} (f_a(U) + |w_3|^2)$  are Ricci-flat Kähler metrics on  $K_{\mathbb{C}P^1} \times E$ . Note that we need to extend these metrics under blowup. These metrics are asymptotically flat at  $\infty$ .

Now we construct the approximate Ricci-flat metrics on  $M$ . We glue  $M_0$  with 8 copies of  $K_{\mathbb{C}P^1} \times E_2$  and 8 copies of  $K_{\mathbb{C}P^1} \times E_3$ , by patching the boundaries of some fixed tubular neighborhoods of  $F$

in  $M_0$ , the boundaries of some fixed tubular neighborhoods of  $\mathbb{C}P^1 \times E_2$  in  $K_{\mathbb{C}P^1} \times E_2$  and the boundaries of some fixed tubular neighborhoods of  $\mathbb{C}P^1 \times E_3$  in  $K_{\mathbb{C}P^1} \times E_3$ . Topologically the result manifold is  $M$ . We glue the Kähler potential of the flat orbifold metric on  $M_0$  and the 16 copies of the Kähler potential described above to get a function  $h_{\vec{a}}$  (we use different  $a$ 's for different divisors,  $\vec{a} = (a_1, \dots, a_{16})$ ). Since all metrics are nearly flat near the boundaries as long as  $\vec{a}$  is very small,  $g_{\vec{a}} = \sqrt{-1}\partial\bar{\partial}h_{\vec{a}}$  are Kähler metrics on  $M$ . They are very close to Ricci flat (as close as we want by choosing small  $\vec{a}$ ). From Yau's existence theorem of Ricci-flat Kähler metrics (see [Y]), there is a unique function  $u_{\vec{a}}$  on  $M$  with  $\int_M u_{\vec{a}} dg_{\vec{a}} = 0$  such that  $g_{\vec{a}}^{RF}$  is Ricci flat, where  $g_{\vec{a}}^{RF} = g_{\vec{a}} + \sqrt{-1}\partial\bar{\partial}u_{\vec{a}}$ .

We need the following

**Theorem 2.** *Let  $F$  be the set of singular points in  $M_0$ . For any relatively compact set  $W$  in the complement of the proper transformation of  $F$  in  $M$ , there exists positive constant  $C$  independent of  $\vec{a}$  but depending on  $W$ , such that*

$$(2) \quad \|u_{\vec{a}}\|_{\tilde{C}^{4,\alpha}(W)} \leq C \cdot |\vec{a}|^2,$$

where  $\tilde{C}^{4,\alpha}(W)$  is the Hölder norm with respect to some fixed coordinate system on  $M$ .

*Proof.* The approximate Ricci-flat metrics  $g_{\vec{a}}$  on  $M$  are similar to the approximate Ricci-flat metrics  $\omega_a$  in [L]. The same proof of Theorem 3.4 in [L] gives the proof of the theorem here.  $\square$

**1c.** Special Lagrangian tori on  $M$ . First  $E_1 \times E_2 \times E_3$  has special Lagrangian torus fibration with respect to holomorphic (3,0)-form  $dz_1 \wedge dz_2 \wedge dz_3$  and the flat metric, namely  $T_{\alpha,\beta,\gamma} = T_{\alpha} \times T_{\beta} \times T_{\gamma}$  for any real numbers  $\alpha, \beta$  and  $\gamma$ , where  $T_{\alpha} \subset E_1$  is the image of  $\alpha + i\mathbb{R}$  under the projection  $\mathbb{C} \rightarrow E_1$  (Here we need the periods  $\tau_i$  to be pure imaginary,  $i = 1, 2, 3$ ). For generic values of  $\alpha, \beta$  and  $\gamma$ , the image of  $T_{\alpha,\beta,\gamma}$  in  $M_0$  does not intersect with the singular set  $F$ . They are embedded tori in  $M_0$ . Now we conclude that these tori can be perturbed to special Lagrangian tori in  $M$  (embedded).

**Theorem 3.** *Any special Lagrangian torus  $f_0$  in  $M_0$  as described above, can be perturbed to a special Lagrangian torus in  $M$ .*

*Proof.* Assume that open set  $U$  contains the image of  $f_0$  in  $M_0$ , the closure  $\bar{U}$  is compact. On  $\bar{U}$ , the metric  $g_{\vec{a}}^{RF}$  differs from the flat metric on  $E_1 \times E_2 \times E_3$  by an exact form. The difference is small on  $\bar{U}$  by Theorem 2.  $\Omega$  on  $\pi^{-1}(\bar{U})$  is the same as  $\Omega_0$  on  $\bar{U}$ . The image of  $f_0$  in  $M$  is approximate special Lagrangian torus. Now we can apply the proof of Theorem 2.1 in [L] to conclude that  $f_0$  can be perturbed to a special Lagrangian torus provided we choose  $\vec{a}$  small enough.  $\square$

Next we remove the assumption that the periods  $\tau_i$  being pure imaginary. We can view the threefolds with general  $\tau_i$  as deformations of those with pure imaginary periods. Then by applying Theorem 2.1 i) in [L], we conclude that the threefolds have a family of embedded special Lagrangian tori when the real parts of  $\tau_i$  are sufficiently small.

We check that the special Lagrangian torus  $f : T^3 \rightarrow M$  satisfies  $f^*H^2(M) = 0$ . This is the condition required by mirror symmetry (see [L]). Note that  $h^2(M) = 19$  by [B]. The following are

a basis of  $H^2(M)$ . There are 16 classes which are the Poincare dual of the exceptional divisors in  $M$ . Since the image  $f(T^3)$  has no intersection with the exceptional divisors, the pull-backs of these classes are zero. Again let  $z_1 = x_1 + iy_1$ ,  $z_2 = x_2 + iy_2$ ,  $z_3 = x_3 + iy_3$  be the complex coordinates of  $E_1 \times E_2 \times E_3$ . Then  $dx_1 \wedge dy_1$ ,  $dx_2 \wedge dy_2$ ,  $dx_3 \wedge dy_3$  are the classes of degree two invariant under actions  $\alpha$  and  $\beta$ . They can be lifted to three classes in  $H^2(M)$ . Obviously their pull-backs on  $T^3$  are zero classes.

## 2. SPECIAL LAGRANGIAN SUBMANIFOLDS ON $K_{CP^n}$

In the following the periods  $\tau_1$ ,  $\tau_2$  and  $\tau_3$  are pure imaginary. From section 1c we know that one can perturb any special Lagrangian torus  $T_{\alpha,\beta,\gamma}$  to one in  $M$  as long as  $\vec{a}$  is small, except  $(\alpha, \beta) = (\frac{1}{4}, \frac{1}{4})$ ,  $(\frac{1}{4}, \frac{3}{4})$ ,  $(\frac{3}{4}, \frac{1}{4})$ ,  $(\frac{3}{4}, \frac{3}{4})$  or  $(\alpha, \gamma) = (0, 0)$ ,  $(0, \frac{1}{2})$ ,  $(\frac{1}{2}, 0)$ ,  $(\frac{1}{2}, \frac{1}{2})$  (these are also tori which are preserved by either action  $\alpha$  or  $\beta$ ). Note that each such torus intersects 4 fixed elliptic curves in  $F$ . For example,  $T_{0,\beta,0}$  intersects  $0 \times E_2 \times 0$ ,  $0 \times E_2 \times \frac{\tau_3}{2}$ ,  $\frac{\tau_1}{2} \times E_2 \times 0$ ,  $\frac{\tau_1}{2} \times E_2 \times \frac{\tau_3}{2}$ . Here we try to describe what happens to these special Lagrangian tori.

Let  $z_1 = u_1 + iv_1$ ,  $z_2 = u_2 + iv_2$  be coordinate on  $\mathbb{C}^2$ . A obvious family of special Lagrangian submanifolds  $L_{bc}^0$  on  $\mathbb{C}^2$  are given by:  $u_1 + iu_2 = (b + ic)(v_2 + iv_1)$ . It is clear that they are invariant under the  $\mathbb{Z}_2$  action of  $\mathbb{C}^2$ . We show

**Theorem 4.** *Under the blow up of  $\mathbb{C}^2/\mathbb{Z}_2$  (the blowup is  $K_{CP^1}$ ),  $L_{bc}^0$  give a family of special Lagrangian which covers  $K_{CP^1}$ .*

*Proof.* First we show that there are submanifolds  $L_{bc}$  in  $K_{CP^1}$  corresponding to extending  $L_{bc}^0$  across the exceptional divisor. Blowup coordinates are  $p = p_1 + ip_2$ ,  $q = q_1 + iq_2$  with  $z_1 = p^{1/2}$ ,  $z_2 = p^{1/2}q$ . Combining with the equation of  $L_{bc}^0$  we get by eliminating  $u_1, u_2, v_1, v_2$

$$(3) \quad bq_1^2 + bq_2^2 - 2bq_1 + (b^2 + c^2 - 1)q_2 - b = 0,$$

$$(4) \quad \frac{(bq_1 - c)^2 - (bq_2 - 1)^2}{(bq_1 - c)^2 + (bq_2 - 1)^2} = \frac{p_1}{\sqrt{p_1^2 + p_2^2}}.$$

Equation (3) shows that the intersection of special Lagrangian  $L_{bc}$  with  $CP^1$  is a circle. So the topological type of  $L_{bc}$  is  $S^1 \times \mathbb{R}$  for all  $b$  and  $c$ . Note that  $L_{bc}$  cover  $CP^1$  for  $c = 0$  and  $-1 \leq b \leq 1$ . It is not a fibration and there is no sub-family of  $L_{bc}$  to form a fibration of  $CP^1$  with fibre  $L_{bc} \cap CP^1$ .

Similar to **1a** the holomorphic (2,0)-form  $\Omega$  on  $K_{CP^1}$  is induced from  $dz_1 \wedge dz_2$ . So  $\text{Im}\Omega|_{L_{bc}} = 0$ .

The Ricci-flat metric on  $K_{CP^1}$  is given by

$$\omega = \frac{\sqrt{-1}}{U^2 \sqrt{1 + U^2}} [(1 + U^2)U \partial \bar{\partial} U - \partial U \wedge \bar{\partial} U].$$

where  $U = |z_1|^2 + |z_2|^2$ . A easy calculation shows that  $\partial \bar{\partial} U|_{L_{bc}} = 0$  and  $\partial U|_{L_{bc}}$  is real one form. So  $\omega|_{L_{bc}} = 0$ .  $L_{bc}$  are special Lagrangian submanifolds.  $\square$

Note that special Lagrangian  $L_{00}^0 \times T_\beta$  matches with special Lagrangian  $T_{0\beta 0}$  near each singular locus  $0 \times E_2 \times 0$ ,  $0 \times E_2 \times \frac{\tau_3}{2}$ ,  $\frac{\tau_1}{2} \times E_2 \times 0$ , and  $\frac{\tau_1}{2} \times E_2 \times \frac{\tau_3}{2}$  in  $M_0$ . We glue four copies of  $L_{00} \times T_\beta$  to  $T_{0\beta 0}$  and get  $\tilde{T}_{0\beta 0}$  in  $M$ , it is tempting to think that some perturbation of  $\tilde{T}_{0\beta 0}$  will be a special Lagrangian torus in  $M$ . We can not prove it because we do not know how to estimate the first eigenvalue of Laplace operator acting on  $\Omega^1(\tilde{T}_{0\beta 0})$ .

Finally we construct special Lagrangian submanifolds in  $K_{\mathbb{C}P^{n-1}}$  which is isomorphic to the blowup of  $\mathbb{C}^n/\mathbb{Z}_n$ . Let  $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$  be complex coordinates of  $\mathbb{C}^n$ . The holomorphic  $(n, 0)$ -form  $\Omega$  on  $K_{\mathbb{C}P^{n-1}}$  is the pull-back of  $dz_1 \wedge \dots \wedge dz_n$ . Let  $U = |z_1|^2 + \dots + |z_n|^2$ . The Ricci-flat Kähler form on  $K_{\mathbb{C}P^{n-1}}$  is

$$\omega = \sqrt{-1}(f'(U)\partial\bar{\partial}U + f''(U)\partial U \wedge \bar{\partial}U),$$

$$f'(U) = (1 + \frac{1}{U^n})^{1/n}.$$

Let  $\vec{x} = (x_1, \dots, x_n)$ ,  $\vec{y} = (y_1, \dots, y_n)$  and  $A = (a_{ij})_{n \times n}$  a real matrix. Consider the real  $n$ -dimensional plane  $L_A^0$  in  $\mathbb{C}^n$  defined by  $\vec{x} = A\vec{y}$ . It is easy to check that  $\partial\bar{\partial}U|_{L_A^0} = 0$  and  $\partial U|_{L_A^0}$  is real one form if and only if  $A$  equals its transpose  $A^t$ . So  $A = A^t$  implies that  $\omega|_{L_A^0} = 0$ . Note that the only singular point on the image of  $L_A^0$  in  $\mathbb{C}^n/\mathbb{Z}_n$  is origin.

Let  $z_1 = w_1^{1/n}$ ,  $z_2 = w_1^{1/n}w_2$ ,  $\dots$ ,  $z_n = w_1^{1/n}w_n$  be the blowup coordinates. Let  $L_A$  be the image of  $L_A^0$  under blowup. We show that  $L_A$  is smooth. Rewrite  $\vec{x} = A\vec{y}$  in terms of  $z_1 = x_1 + iy_1$ ,  $w_2 = u_2 + iv_2$ ,  $\dots$ ,  $w_n = u_n + iv_n$ , we have

$$(5) \quad (1 - \sum_{j=2}^n a_{1j}v_j)x_1 = (a_{11} + \sum_{j=2}^n a_{1j}u_j)y_1,$$

$$(6i) \quad (u_i - \sum_{j=2}^n a_{ij}v_j)x_1 = (a_{i1} + v_i + \sum_{j=2}^n a_{ij}u_j)y_1, \quad 2 \leq i \leq n.$$

Dividing (6i) by (5) we get  $n - 1$  equations defining the intersection of  $L_A$  with  $\mathbb{C}P^{n-1}$  which is smooth. In particular when  $n = 3$ ,  $a_{11} = a_{22} = 1$  and all other  $a_{ij} = 0$ , we get the equations  $u_2 + v_2 = 0$ ,  $u_3 + v_3 = 0$  which defines a  $S^1 \times S^1$  in  $\mathbb{C}P^2$ . So the topological type of  $L_A$  is  $S^1 \times S^1 \times \mathbb{R}$  for the  $A$ .

Let  $A_{i_1 \dots i_k}$  be a  $k \times k$  matrix formed from elements in  $A$  with both row and column numbers in set  $\{i_1, \dots, i_k\}$ . Then

$$dz_1 \wedge \dots \wedge dz_n|_{L_A^0} = (\sum_{k=\text{even}} \sum_{i_1 < \dots < i_k} i^k \det(A_{i_1 \dots i_k}) + \sum_{k=\text{odd}} \sum_{i_1 < \dots < i_k} i^k \det(A_{i_1 \dots i_k})) dy_1 \wedge \dots \wedge dy_n.$$

Taking the phase factor into consideration we conclude that  $L_A$  is special Lagrangian submanifolds in  $K_{\mathbb{C}P^{n-1}}$  when  $A = A^t$  and

$$\sin \theta \cdot \sum_{k=\text{even}} \sum_{i_1 < \dots < i_k} \det(A_{i_1 \dots i_k}) + \cos \theta \sum_{k=\text{odd}} \sum_{i_1 < \dots < i_k} \det(A_{i_1 \dots i_k}),$$

where  $0 \leq \theta < 2\pi$ . Furthermore any real  $n$ -dimensional hyperplane in  $\mathbb{C}^n$ , which is the limit of special Lagrangian  $L_A^0$ , gives rise to a special Lagrangian submanifold in  $K_{\mathbb{C}P^{n-1}}$ .

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